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Quantum Langevin equation: a quadratic system

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Abstract. A system described by a quadratic Hamiltonian with a two-photon absorption and emission term is coupled to heat bath, described by either a non-Hermitian or a Hermitian noise operator. The dissipation kernel is memory dependent. A set of dispersion relations for the bath distribution functions is obtained. For the near-Markovian case an explicit solution of the bath distribution function is obtained perturbatively. The KMS condition on the two-point correlators is obtained at the equilibrium limit. The equilibrium system number density is also evaluated.

1. Introduction

An open quantum system with a small number of degrees of freedom interacting weakly and linearly with a dissipative environment described by a heat bath representing an infinite number of degrees of freedom is of longstanding interest (Ford *et al* 1965). The full Hamiltonian equations, after a series of plausible approximations (Haken 1970) generate the quantum analogues of the classical Langevin equations. A quantum Langevin for a harmonic oscillator interacting with a heat bath was recently studied by Streater (1982) and Hasegawa *et al* (1985). The key consistency requirement adopted by these authors is the validity of the canonical commutation relation for the harmonic oscillator dynamical variables for all time $t > 0$. This necessitates a quantum treatment of the heat bath because otherwise the harmonic oscillator variables would violate the canonical commutation relation. Streater (1982) adopted a model for the heat bath comprising of the positive frequency modes alone; and in the Markovian case characterised by an instantaneous damping kernel, he determined the frequency distribution of the heat bath oscillators. He also dynamically established the KMS periodicity conditions (Kubo 1957, Martin and Schwinger 1959) for the Green functions in the equilibrium limit. A shortcoming in the above construction is that the density of the heat bath frequency modes depend on an arbitrary indeterminate function. To remedy this deficiency, Hasegawa *et al* (1985) introduced a quantum heat bath of Hermitian character. Exploiting the consistency conditions mentioned earlier, they expressed the heat bath spectrum density as an universal linear function of the mode frequency multiplied by a factor depending on the system parameters.

We consider a two-fold variation of this problem. Instead of a harmonic oscillator, we consider a general quadratic system coupled to heat bath. This is of relevance in the context of the radiation processes involving a two-photon emission or absorption (Yuen 1976). The system Hamiltonian is given by

$$H_s = \hbar\omega a^\dagger a + \tfrac{1}{2}\hbar(Da^{\dagger 2} + D^* a^2) \tag{1.1}$$

where D is the coupling constant for the two-photon emission process. It is well known that, using a linear Bogoliubov transformation, the Hamiltonian (1.1) can be diagonalised

$$H_s = \hbar\Omega A^\dagger A - \tfrac{1}{2}\hbar(\omega - \Omega) \tag{1.2}$$

where

$$A = \mu a + \nu a^\dagger \tag{1.3}$$

$$\mu = D^*[2\Omega(\omega - \Omega)]^{-1/2}\,\mathrm{e}^{\mathrm{i}\psi} \tag{1.4}$$

$$\nu = |D|[2\Omega(\omega + \Omega)]^{-1/2}\,\mathrm{e}^{\mathrm{i}\psi} \tag{1.5}$$

$$\Omega = (\omega^2 - |D|^2)^{1/2}. \tag{1.6}$$

The system bath coupling is, however, inequivalent in two alternate pictures. We describe the coupling in terms of the coordinate a, in contrast to the transformed coordinate A. We will comment on this later. We assume (Chakrabarti and Vasudevan 1988) a non-Markovian arbitrarily memory-dependent dissipation kernel. We consider both non-Hermitean and Hermitian noise. In these cases it is possible to determine the bath frequency distribution as a perturbation series, where the leading term corresponds to a Markovian limit. A complete set of consistency conditions for the bath frequency distribution function can be established non-perturbatively. The KMS periodicity condition can be derived in the equilibrium limit. We also obtain the equilibrium number density of the system.

In the non-Hermitian case the system bath coupling is taken to be in the 'random wave approximation' (RWA) scheme in that it neglects the rapidly oscillating terms in the interaction. When viewed in terms of the transformed coordinate A, the coupling becomes of the general linear type and its RWA character is lost. This marks an essential difference between the present problem and the previous works (Streater 1982, Chakrabarti and Vasudevan 1988). The distribution function for the heat bath frequency modes depend on an arbitrary undetermined function for the whole frequency range. In the untraviolet region the distribution function asymptotically reaches a constant value. This property is markedly different from that of the simple harmonic oscillator system, where the bath distribution function is constant except in the low-frequency region, where it depends on an indeterminate function.

On the other hand, for a heat bath described by a Hermitian noise operator, the frequency distribution is a universal linear function apart from a system-dependent multiplying factor. Here the presence of the D term introduces a simple scale factor and the linear behaviour of the bath mode distribution remains unaltered. This can be easily understood. A Bogoliubov transformation along with a simple rescaling of the bath noise operator reduces the Langevin equation for this case to that of a harmonic oscillator, studied by Hasegawa *et al* (1985). In the Markovian limit our distribution agrees with their result except for a scaling factor. Here our contribution is a perturbative solution of the bath frequency distribution in the presence of a memory-dependent dissipation kernel.

The plan of the paper is as follows. Sections 2 and 3 contain our description of the quadratic system in the presence of a non-Hermitian and a Hermitian bath, respectively. In section 4 we establish the KMS periodicity condition for both types of baths. We also find the number density of the system in the equilibrium limit. We conclude in section 5.

2. Non-Hermitian heat bath

The Langevin equation for a general quadratic system in the presence of a dissipation kernel $\gamma(t)$ is

$$\dot{a}(t)+\mathrm{i}\omega a(t)+\mathrm{i}Da^{\dagger}(t)+\int_0^t \gamma(t-s)a(s)\,\mathrm{d}s=b(t) \tag{2.1}$$

where $b(t)$ is taken to be a non-Hermitian bath annihilation operator coupled to the system in the RWA mechanism. An alternative starting point is to consider the Langevin equation for the Bogoliubov transformed operator $A(t)$:

$$\dot{A}(t)+\mathrm{i}\Omega A(t)+\int_0^t \gamma(t-s)A(s)\,\mathrm{d}s=\mu b(t)+\nu b^{\dagger}(t). \tag{2.2}$$

The heat bath noise operator in (2.2) reflects a system bath interaction Hamiltonian containing terms of the type $A^{\dagger}b^{\dagger}$ and its Hermitian adjoint. These terms spoil the RWA structure of the interaction Hamiltonian. Here we will retain the description (2.1) of the system and solve for the density of the bath frequency modes.

The bath noise operator $b(t)$ has an expansion in terms of the positive frequency modes (Streater 1982):

$$b(t)=\int_0^{\infty}\mathrm{d}k\,f(k)b(k)\,\mathrm{e}^{-\mathrm{i}t}. \tag{2.3}$$

The operators $b(k)$ satisfy the canonical commutation relation

$$[b(k),b^{\dagger}(k')]=\delta(k-k'). \tag{2.4}$$

Considering the Laplace transform of (2.1) and its Hermitian adjoint, we obtain

$$\begin{pmatrix}\tilde{a}(s)\\ \tilde{a}^{\dagger}(s)\end{pmatrix}=(\det\mathbf{M})^{-1}\mathbf{M}\begin{pmatrix}a(0)+\tilde{b}(s)\\ a^{\dagger}(0)+\tilde{b}^{\dagger}(s)\end{pmatrix} \tag{2.5}$$

where

$$\mathbf{M}=\begin{pmatrix}S-\mathrm{i}\omega+\tilde{\gamma}^*(s) & -\mathrm{i}D\\ \mathrm{i}D^* & S+\mathrm{i}\omega+\tilde{\gamma}(s)\end{pmatrix} \tag{2.6}$$

and the Laplace transform $\tilde{F}(s)$ of a construct $F(t)$ is defined as

$$\tilde{F}(s)=\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-st}F(t). \tag{2.7}$$

A typical memory-dependent kernel with a strength γ and a timescale for memory τ is

$$\gamma(t)=(\gamma/\tau)\theta(t)\exp(-t/\tau). \tag{2.8}$$

For this choice (2.5) may be easily inverted to obtain $a(t)$:

$$a(t)=\alpha(t)a(0)+\beta(t)a^{\dagger}(0)+\int_0^t \alpha(t-s)b(s)\,\mathrm{d}s+\int_0^t \beta(t-s)b^{\dagger}(s)\,\mathrm{d}s \tag{2.9}$$

where the admittance kernels $\alpha(t)$ and $\beta(t)$ are given by

$$\alpha(t)=\sum_i c_i\exp(-\mathrm{i}\Omega_i t-\Gamma_i t) \tag{2.10}$$

$$\beta(t)=\sum_i d_i\exp(-\mathrm{i}\Omega_i t-\Gamma_i t). \tag{2.11}$$

The boundary conditions are

$$\sum_i c_i = 1 \qquad \sum_i d_i = 0. \tag{2.12}$$

Close to the Markovian limit, i.e. in the regime $\gamma\tau \ll 1$, the constants of the admittance kernels take the form

$$\begin{aligned}
&c_1 \simeq \frac{\omega+\Omega}{2\Omega}(1+\gamma\tau) && d_1 \simeq \frac{D}{2\Omega}(1+\gamma\tau)\\
&\Omega_1 \simeq \Omega(1+\gamma\tau) && \Gamma_1 = \gamma(1+\gamma\tau)\\
&c_2 \simeq -\frac{\omega+\Omega}{2\Omega}\gamma\tau && d_2 \simeq -\frac{D}{2\Omega}\gamma\tau\\
&\Omega_2 \simeq -\Omega\gamma\tau && \Gamma_2 = \frac{1}{\tau} - \gamma(1+\gamma\tau)\\
&c_3 \simeq -\frac{\omega-\Omega}{2\Omega}(1+\gamma\tau) && d_3 \simeq -\frac{D}{2\Omega}(1+\gamma\tau)\\
&\Omega_3 = -\Omega_1 && \Gamma_3 = \Gamma_1\\
&c_4 \simeq \frac{\omega-\Omega}{2\Omega}\gamma\tau && d_4 \simeq \frac{D}{2\Omega}\gamma\tau\\
&\Omega_4 = -\Omega_2 && \Gamma_4 = \Gamma_2.
\end{aligned} \tag{2.13}$$

The requirement to be satisfied is

$$[a(t), a^{\dagger}(t)] = 1 \qquad \text{for } t > 0 \tag{2.14}$$

where $[a(0), a^{\dagger}(0)] = 1$. Substituting (2.9) and its conjugate in (2.14), we obtain

$$|\alpha(t)|^2 - |\beta(t)|^2 + \int_0^{\infty} \mathrm{d}k\rho(k)(|\alpha(k,t)|^2 - |\beta(-k,t)|^2) = 1 \tag{2.15}$$

where

$$\rho(k) = |f(k)|^2 \tag{2.16}$$

$$\alpha(k,t) = \int_0^t \mathrm{d}s\alpha(s)\,\mathrm{e}^{\mathrm{i}ks} \tag{2.17}$$

$$\beta(k,t) = \int_0^t \mathrm{d}s\,\beta(s)\,\mathrm{e}^{\mathrm{i}ks}. \tag{2.18}$$

Considering the limit $t\to\infty$, which is to be understood as $t \gg \gamma^{-1}$, we obtain the time-independent part of (2.15):

$$\begin{aligned}
\sum_{ij}\int_0^{\infty} \mathrm{d}k\,\rho(k)[&c_i c_j^*(k-\Omega_i+\mathrm{i}\Gamma_i)^{-1}(k-\Omega_j-\mathrm{i}\Gamma_j)^{-1}\\
&-d_i d_j^*(k+\Omega_i-\mathrm{i}\Gamma_i)^{-1}(k+\Omega_j+\mathrm{i}\Gamma_j)^{-1}] = 1.
\end{aligned} \tag{2.19}$$

The time-dependent part of (2.15) yields

$$\begin{aligned}
\sum_{ij}\{&(c_i c_j^* - d_i d_j^*)\exp[-\mathrm{i}(\Omega_i-\Omega_j)t - (\Gamma_i+\Gamma_j)t]\\
&+c_i c_j^* F_\rho(\Omega_i,\Omega_j;\Gamma_i,\Gamma_j;t) - d_i d_j^* F_\rho^*(-\Omega_i,-\Omega_j;\Gamma_i,\Gamma_j;t)\} = 0
\end{aligned} \tag{2.20}$$

where

$$F_\rho(\Omega_i, \Omega_j; \Gamma_i, \Gamma_j; t)$$
$$= \int_0^\infty dk\, \rho(k)(k-\Omega_i+i\Gamma_i)^{-1}(k-\Omega_j-i\Gamma_j)^{-1}$$
$$\times \{\exp[-i(\Omega_i-\Omega_j)t-(\Gamma_i+\Gamma_j)t]-\exp[i(k-\Omega_i)t-\Gamma_i t]$$
$$-\exp[-i(k-\Omega_j)t-\Gamma_j t]\}. \quad (2.21)$$

After a contour integration, (2.20) takes the form

$$\sum_{ij} \{c_i c_j^* F\rho(\Omega_i, \Omega_j; \Gamma_i, \Gamma_j) - d_i d_j^* F_\rho^*(-\Omega_i, -\Omega_j; \Gamma_i, \Gamma_j)\}$$
$$\times \exp[-i(\Omega_i-\Omega_j)t-(\Gamma_i+\Gamma_j)t]=0 \quad (2.22)$$

where

$$F_\rho(\Omega_i, \Omega_j; \Gamma_i, \Gamma_j)$$
$$= 1+\int_0^\infty dk\, \rho(k)(k-\Omega_i+i\Gamma_j)^{-1}(k-\Omega_j-i\Gamma_j)^{-1}$$
$$-2\pi i[\Omega_j-\Omega_i+i(\gamma_i+\Gamma_j)]^{-1}[\rho(\Omega_i-i\Gamma_i)+\rho(\Omega_j+i\Gamma_j)]. \quad (2.23)$$

Equating the coefficients of $\exp[-i(\Omega_i-\Omega_j)t-(\Gamma_i+\Gamma_j)t]$ on both sides of (2.22) leads to a set of dispersion relations for all i and j:

$$c_i c_j^* F_\rho(\Omega_i, \Omega_j; \Gamma_i, \Gamma_j) - d_i d_j^* F_\rho^*(-\Omega_i, -\Omega_j; \Gamma_i, \Gamma_j) = 0. \quad (2.24)$$

The dispersion relation (2.24) is valid for an arbitrary memory-dependent kernel and does not depend on the approximation scheme adopted in the remaining part of this section.

An explicit solution for $\rho(k)$ is also possible in the limit $\gamma\tau \ll 1$. For this purpose, we use the expansion

$$\rho(k) = N_0(k) + \gamma\tau N_1(k) + O(\gamma\tau)^2. \quad (2.25)$$

We also assume $N_l(k<0)=0$ for all l. We substitute the expansion (2.25) into the time-independent relation (2.19) and equate the terms $O(l)$ and $O(\gamma\tau)$ separately. Terms $O(l)$ yield

$$\int_0^\infty dk\, N_0(k)\{R_+[(k-\Omega_1)^2+\Gamma_1^2]^{-1}+R_-[(k+\Omega_1)^2+\Gamma_1^2]^{-1}\}=1 \quad (2.26)$$

where

$$R_\pm = \frac{(\omega\pm\Omega)^2-|D|^2}{4\Omega^2}. \quad (2.27)$$

Equating the terms $O(\gamma\tau)$, we obtain

$$\int_0^\infty dk\, N_1(k)\{R_+[(k-\Omega_1)^2+\Gamma_1^2]^{-1}+R_-[(k+\Omega_1)^2+\Gamma_1^2]^{-1}\}$$
$$= -2+R_+\int_0^\infty dk\, N_0(k)[(k-\Omega_1+i\Gamma_1)^{-1}(k-\Omega_2-i\Gamma_2)^{-1}+cc]$$
$$+R_-\int_0^\infty dk\, N_0(k)[(k+\Omega_1+i\Gamma_1)^{-1}(k+\Omega_2-i\gamma_2)^{-1}+cc]. \quad (2.28)$$

On substitution of (2.25) in (2.20) we get after equating terms $O(l)$ and $O(\gamma\tau)$ separately,

$$\begin{aligned}\int_0^\infty \mathrm{d}k\, N_0(k)\{&R_+[(k-\Omega_1)^2+\Gamma_1^2]^{-1}\cos(k-\Omega_1)t\\ &+R_-[(k+\Omega_1)^2+\Gamma_1^2]^{-1}\cos(k+\Omega_1)t\}\\ &=\exp(-\Gamma_1 t)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\int_0^\infty \mathrm{d}k\, N_1(k)\{&R_+[(k-\Omega_1)^2+\Gamma_1^2]^{-1}\cos(k-\Omega_1)t\\ &+R_-[(k+\Omega_1)^2+\Gamma_1^2]^{-1}\cos(k+\Omega_1)t\}\\ &=\Phi(\Omega_1,\Omega_2;\Gamma_1,\Gamma_2)\exp(-\Gamma_1 t)-\tfrac{1}{2}\exp[-\mathrm{i}(\Omega_1-\Omega_2)t-\Gamma_2 t]\\ &\quad\times\{R_+F_{N_0}(\Omega_1,\Omega_2;\Gamma_1,\Gamma_2)+R_-F^*_{N_0}(-\Omega_1,-\Omega_2;\Gamma_1,\Gamma_2)\}\\ &\quad-\tfrac{1}{2}\exp[\mathrm{i}(\Omega_1-\Omega_2)t-\Gamma_2 t]\\ &\quad\times\{R_+F^*_{N_0}(\Omega_1,\Omega_2;\Gamma_1,\Gamma_2)+R_-F_{N_0}(-\Omega_1,-\Omega_2;\Gamma_1,\Gamma_2)\}\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}&\Phi(\Omega_1,\Omega_2;\Gamma_1,\Gamma_2)\\ &\quad=-1+R_+\int_0^\infty \mathrm{d}k\, N_0(k)[(k-\Omega_1+\mathrm{i}\Gamma_1)^{-1}(k-\Omega_2-\mathrm{i}\Gamma_2)^{-1}+\mathrm{cc}]\\ &\quad\quad+R_-\int_0^\infty \mathrm{d}k\, N_0(k)[(k+\Omega_1+\mathrm{i}\Gamma_1)^{-1}(k+\Omega_2-\mathrm{i}\Gamma_2)^{-1}+\mathrm{cc}].\end{aligned} \tag{2.31}$$

To obtain (2.29) and (2.30), we exploited (2.26) and (2.28), respectively. We also carried out a contour integration to obtain the right-hand side of (2.30) in the present form. The coefficients of the terms $\sim\exp(-\Gamma_2 t)$ on the RHS of (2.30) as can be seen by making a perturbative expansion of the dissipation relations (2.24). This is crucial and allows us to determine $N_1(k)$ using the inverse cosine transform. Equation (2.30) reduces to

$$\begin{aligned}\int_0^\infty \mathrm{d}k\, N_1(k)\{&R_+[(k-\Omega_1)^2+\Gamma_1^2]^{-1}\cos(k-\Omega_1)t\\ &+R_-[(k+\Omega_1)^2+\Gamma_1^2]^{-1}\cos(k+\Omega_1)t\}\\ &=\Phi(\Omega_1,\Omega_2,\Gamma_1,\Gamma_2)\exp(-\Gamma_1 t).\end{aligned} \tag{2.32}$$

To obtain an explicit solution for the leading-order (Markovian) distribution function, we use the standard cosine transform for $\exp(-\Gamma_1 t)$ in (2.29). This leads to

$$R_+[N_0(k+\Omega_1)+N_0(-k+\Omega_1)]+R_-N_0(k-\Omega_1)=2\Gamma_1/\pi. \tag{2.33}$$

The general solution of (2.33) is

$$\begin{aligned}N_0(2n\Omega_1+x)|_{n\geqslant 0}&=\frac{2\Gamma_1}{\pi}\left[1-\left(\frac{\omega-\Omega}{\omega+\Omega}\right)^n\right]+\left(\frac{\omega-\Omega}{\omega+\Omega}\right)^n\sigma_0(x)\\ N_0((2n+1)\Omega_1+x)|_{n\geqslant 0}&=\frac{2\Gamma_1}{\pi}\left[1-\left(\frac{\omega-\Omega}{\omega+\Omega}\right)^{n+1}\right]-\left(\frac{\omega-\Omega}{\omega+\Omega}\right)^n\sigma_0(\Omega_1-x)\end{aligned} \tag{2.34}$$

where $0 \leqslant x \leqslant \Omega_1$ and the arbitrary positive function $\sigma_0(x)$ has the property

$$\begin{aligned}
&\sigma_0(k \leqslant 0) = 0 \\
&\sigma_0(\Omega_1) = \frac{\Gamma_1}{\pi} \frac{2\Omega}{\omega + \Omega} \\
&\sigma_0(2\Omega_1) = \frac{2\Gamma_1}{\pi} \frac{2\Omega}{\omega + \Omega}.
\end{aligned} \tag{2.35}$$

An essential difference from the case of the harmonic oscillator coupled to the bath in the RWA mechanism (Streater 1982) is that here the distribution function depends on the arbitrary function $\sigma_0(x)$ for all values of k. In the large-k ($\gg \Omega_1$) limit, we obtain from (2.34)

$$N_0(k \gg \Omega_1) = \frac{2\Gamma_1}{\pi}. \tag{2.36}$$

The first-order distribution function $N_1(k)$ is obtained in an analogous way. We enlist

$$\begin{aligned}
&N_1(2n\Omega_1 + x)|_{n \geqslant 0} \\
&\quad = \frac{2\Gamma_1}{\pi} \Phi(\Omega_1, \Omega_2, \Gamma_1, \Gamma_2) \left[1 - \left(\frac{\omega - \Omega}{\omega + \Omega}\right)^n\right] + \left(\frac{\omega - \Omega}{\omega + \Omega}\right)^n \sigma_1(x) \\
&N_1((2n+1)\Omega_1 + x)|_{n \geqslant 0} \\
&\quad = \frac{2\Gamma_1}{\pi} \Phi(\Omega_1, \Omega_2, \Gamma_1, \Gamma_2) \left[1 - \left(\frac{\omega - \Omega}{\omega + \Omega}\right)^{n+1}\right] - \left(\frac{\omega - \Omega}{\omega + \Omega}\right)^n \sigma_1(\Omega_1 - x)
\end{aligned} \tag{2.37}$$

where the arbitrary function $\sigma_1(x)$ is not necessarily positive and has the property

$$\begin{aligned}
&\sigma_1(k \leqslant 0) = 0 \\
&\sigma_1(\Omega_1) = \frac{\Gamma_1}{\pi} \frac{2\Omega}{\omega + \Omega} \Phi(\Omega_1, \Omega_2, \Gamma_1, \Gamma_2) \\
&\sigma_1(2\Omega_1) = \frac{2\Gamma_1}{\pi} \frac{2\Omega}{\omega + \Omega} \Phi(\Omega_1, \Omega_2, \Gamma_1, \Gamma_2).
\end{aligned} \tag{2.38}$$

The present expansion scheme with a perturbation parameter $\gamma\tau$ can be developed up to an arbitrary order with the lowest order representing a Markovian process. At each order an undetermined function ($\sigma_0(x)$, $\sigma_1(x)$, etc) has to be introduced to describe the bath distribution function.

3. Hermitian bath

A heat bath described by a Hermitian noise operator $b(t)$ was considered by Hasegawa *et al* (1985). An analysis of the general quadratic system coupled to a Hermitian bath in the presence of an arbitrarily memory-dependent kernel may be developed parallel to section 2. We will discuss the difference between the two cases. The Langevin equations for the quadratic system are

$$\dot{a}(t) + \mathrm{i}\omega a(t) + \mathrm{i}Da^{\dagger}(t) + \int_0^t \gamma(t-s)a(s) = b(t) \tag{3.1}$$

$$\dot{a}^{\dagger}(t) - \mathrm{i}\omega a^{\dagger}(t) - \mathrm{i}D^* a(t) + \int_0^t \gamma(t-s)a^{\dagger}(s) = b(t). \tag{3.2}$$

In terms of the Bogoliubov transformed operator $A(t)$, the Langevin equations (3.1) and (3.2) may be rephrased as

$$\dot{A}(t)+\mathrm{i}\Omega A(t)+\int_0^t \gamma(t-s)A(s)\,\mathrm{d}s=(\mu+\nu)b(t). \tag{3.3}$$

Apart from the rescaling factor $(\mu+\nu)$ for the noise operator, (3.3) differs from the Langevin equation considered by Hasegawa *et al* (1985) by the presence of the memory-dependent kernel $\gamma(t)$. So our results in the present section may be viewed as an extension of the results of Hasegawa *et al* (1985) for an arbitrarily memory-dependent kernel $\gamma(t)$. We also derive a complete set of dispersion relations for the heat bath frequency mode distribution function. To maintain a uniformity with section 2, however, we consider the Langevin equation (3.1) and (3.2) rather than (3.3).

For the dissipation kernel $\gamma(t)$ we can readily write the solution of (3.1):

$$a(t)=\alpha(t)a(0)+\beta(t)a^{\dagger}(0)+\int_0^t [\alpha(t-s)+\beta(t-s)]b(s)\,\mathrm{d}s. \tag{3.4}$$

Using the Fourier expansion of the Hermitian noise operator

$$b(t)=\int_0^{\infty} \mathrm{d}k[f(k)b(k)\,\mathrm{e}^{-\mathrm{i}kt}+f^*(k)b^{\dagger}(k)\,\mathrm{e}^{\mathrm{i}kt}] \tag{3.5}$$

Hasegawa *et al* (1985) derived the commutation relation

$$[b(t),b(s)]=\int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{sgn}(k)\rho(k)\exp[-\mathrm{i}k(t-s)]. \tag{3.6}$$

Following these authors, we define

$$\hat{\rho}(k)=\mathrm{sgn}(k)\rho(k) \tag{3.7}$$

where $\hat{\rho}(k)$ is not a positive definite quantity. The key requirement for the commutation relation (2.14) leads to

$$|\alpha(t)|^2-|\beta(t)|^2+\int_0^{\infty} \mathrm{d}k\hat{\rho}(k)|\alpha(k,t)+\beta(k,t)|^2=1. \tag{3.8}$$

The time-independent equation is

$$\int_{-\infty}^{\infty} \mathrm{d}k\,\hat{\rho}(k)\sum_{ij}(c_i+d_i)(c_j^*+d_j^*)(k-\Omega_i+\mathrm{i}\Gamma_i)^{-1}(k-\Omega_j-\mathrm{i}\Gamma_j)^{-1}=1. \tag{3.9}$$

The time-dependent equation has the form

$$\sum_{ij}(c_ic_j^*-d_id_j^*)\exp[-\mathrm{i}(\Omega_i-\Omega_j)t-(\Gamma_i+\Gamma_j)t.]$$
$$+\sum_{ij}(c_i+d_i)(c_j^*+d_j^*)G_{\hat{\rho}}(\Omega_i,\Omega_j;\Gamma_i,\Gamma_j;t)=0 \tag{3.10}$$

where

$$G_{\hat{\rho}}(\Omega_i,\Omega_j;\Gamma_i,\Gamma_j;t)$$
$$=\int_{-\infty}^{\infty} \mathrm{d}k\,\hat{\rho}(k)(k-\Omega_i+\mathrm{i}\Gamma_i)^{-1}(k-\Omega_j-\mathrm{i}\Gamma_j)^{-1}$$
$$\times\{\exp[-\mathrm{i}(\Omega_i-\Omega_j)t-(\Gamma_i+\Gamma_j)t]-\exp[\mathrm{i}(k-\Omega_i)t-\Gamma_it]$$
$$-\exp[-\mathrm{i}(k-\Omega_j)t-\Gamma_jt]\}. \tag{3.11}$$

By performing a contour integration and equating terms with an identical time behaviour, we write a set of dispersion relations:

$$c_i c_j^* - d_i d_j^* + (c_i + d_i)(c_j^* + d_j^*) G_{\hat{\rho}}(\Omega_i, \Omega_j; \Gamma_i, \Gamma_j) = 0 \tag{3.12}$$

where

$$G_{\hat{\rho}}(\Omega_i, \Omega_j; \Gamma_i, \Gamma_j) = \int_{-\infty}^{\infty} \mathrm{d}k\, \hat{\rho}(k)(k - \Omega_i + \mathrm{i}\Gamma_i)^{-1}(k - \Omega_j - \mathrm{i}\Gamma_j)^{-1} - 2\pi\mathrm{i}(\Omega_j - \Omega_i + \mathrm{i}(\Gamma_i + \Gamma_j))^{-1}[\hat{\rho}(\Omega_i - \mathrm{i}\Gamma_i) + \hat{\rho}(\Omega_j + \mathrm{i}\Gamma_j)]. \tag{3.13}$$

To solve explicitly for $\hat{\rho}(k)$ we adopt a perturbative expansion in $\gamma\tau$

$$\hat{\rho}(k) = \hat{N}_0(k) + \gamma\tau \hat{N}_1(k) + \mathrm{O}(\gamma\tau)^2 \tag{3.14}$$

and proceed exactly as before. We only quote our results here. The density functions $N_0(k)$ and $N_1(k)$ satisfy the following integral equations:

$$\frac{\omega + \mathrm{Re}(D)}{\Omega} \int_{-\infty}^{\infty} \mathrm{d}k\, \hat{N}_0(k)[(k - \Omega_1)^2 + \Gamma_1^2]^{-1} \cos(k - \Omega_1)t = \exp(-\Gamma_1 t) \tag{3.15}$$

$$\frac{\omega + \mathrm{Re}(D)}{\Omega} \int_{-\infty}^{\infty} \mathrm{d}k\, \hat{N}_1(k)[(k - \Omega_1)^2 + \Gamma_1^2]^{-1} \cos(k - \Omega_1)t = \psi(\Omega_1, \Omega_2, \Gamma_1, \Gamma_2) \exp(-\Gamma_1 t) \tag{3.16}$$

where

$$\psi(\Omega_1, \Omega_2; \Gamma_1, \Gamma_2) = -1 + \frac{\omega + \mathrm{Re}(D)}{\Omega} \int_{-\infty}^{\infty} \mathrm{d}k\, \hat{N}_0(k) \times [(k - \Omega_1 + \mathrm{i}\Gamma_1)^{-1}(k - \Omega_2 - \mathrm{i}\Gamma_2)^{-1} + \mathrm{cc}]. \tag{3.17}$$

The solutions of (3.15) and (3.16) are

$$\hat{N}_0(k) = (\Gamma_1 \Omega / \pi \Omega_1)(\omega + \mathrm{Re}(D))^{-1} k \tag{3.18}$$

and

$$N_1(k) = (\Gamma_1 \Omega / \pi \Omega_1)(\omega + \mathrm{Re}(D))^{-1} \psi(\Omega_1, \Omega_2; \Gamma_1, \Gamma_2) k. \tag{3.19}$$

The corresponding $\rho(k)$ may be read directly from (2.7) and is a linear function of $|k|$ alone.

4. Green functions

The equilibrium two-point Green function for the system dynamical variables satisfies the KMS periodicity condition. We establish the KMS periodicity condition for an arbitrary memory-dependent dissipation kernel $\gamma(t)$. The heat bath may be described by either a non-Hermitian or a Hermitian noise. In this section we do not use the expansion scheme in the parameter $\gamma\tau$ developed earlier.

The ensemble average for the bath frequency modes is given by

$$\langle b^\dagger(k) b(k) \rangle = n(k)\delta(k - k') \tag{4.1}$$

where

$$n(k) = (\exp(\beta k) - 1)^{-1}. \tag{4.2}$$

In (4.1) and hereafter $\langle\ \rangle$ will denote the average over the bath degrees of freedom.

In the non-Hermitian noise case, using (2.9) we determine the system correlation function

$$\begin{aligned}
\langle a^\dagger(T)a(T+t)\rangle \\
&= \alpha^*(T)\alpha(T+t)\langle a^\dagger(0)a(0)\rangle \\
&\quad + \alpha^*(T)\beta(T+t)\langle a^\dagger(0)a^\dagger(0)\rangle \\
&\quad + \beta^*(T)\alpha(T+t)\langle a(0)a(0)\rangle \\
&\quad + \beta^*(T)\beta(T+t)\langle a(0)a^\dagger(0)\rangle \\
&\quad + \int_0^\infty \mathrm{d}k\, n(k)\rho(k)\,\mathrm{e}^{-\mathrm{i}kt}\left(\int_0^T \mathrm{d}\tau_1\, \alpha^*(\tau_1)\,\mathrm{e}^{-\mathrm{i}k\tau_1}\right)\left(\int_0^{T+t} \mathrm{d}\tau_2\, \alpha(\tau_2)\,\mathrm{e}^{\mathrm{i}k\tau_2}\right) \\
&\quad + \int_0^\infty \mathrm{d}k\, (n(k)+1)\rho(k)\,\mathrm{e}^{\mathrm{i}kt}\left(\int_0^T \mathrm{d}\tau_1\, \beta^*(\tau_1)\,\mathrm{e}^{\mathrm{i}k\tau_1}\right)\left(\int_0^{T+t} \mathrm{d}\tau_2\, \beta(\tau_2)\,\mathrm{e}^{-\mathrm{i}k\tau_2}\right).
\end{aligned} \tag{4.3}$$

As $T\to\infty$, the terms depending on the *initial* system variables vanish and the factors in the parenthesis converge to their corresponding Fourier transforms: such as

$$\lim_{T\to\infty}\int_0^T \mathrm{d}\tau\, \alpha(\tau)\,\mathrm{e}^{\mathrm{i}k\tau} = \alpha(k). \tag{4.4}$$

Using these results, we obtain

$$\begin{aligned}
G^<(t) &= \langle a_\infty^\dagger a_\infty(t)\rangle \\
&= \int_0^\infty \mathrm{d}k\,\rho(k)[|\alpha(k)|^2 n(k)\,\mathrm{e}^{-\mathrm{i}kt} + |\beta(-k)|^2(n(k)+1)\,\mathrm{e}^{\mathrm{i}kt}].
\end{aligned} \tag{4.5}$$

Similarly $\langle a(T+t)a^\dagger(T)\rangle$ converge to

$$G^>(t) \equiv \langle a_\infty(t)a_\infty^\dagger\rangle = \int_0^\infty \mathrm{d}k\,\rho(k)[|\alpha(k)|^2(n(k)+1)\,\mathrm{e}^{-\mathrm{i}kt} + |\beta(-k)|^2 n(k)\,\mathrm{e}^{\mathrm{i}kt}]. \tag{4.6}$$

In the limit $T\to\infty$, the Green functions $G^>(y)$ and $G^<(t)$ are translation invariant. From (4.2), (4.5), (4.6) we obtain

$$G^>(t-\mathrm{i}\beta) = G^<(t) \tag{4.7}$$

which is the KMS condition.

The expectation value of the number operator for the system may be obtained at an arbitrary time T:

$$N_T \equiv \langle a^\dagger(T)a(T)\rangle. \tag{4.8}$$

At $T\to\infty$ the number operator becomes independent of the initial density matrix of the system and is

$$N_\infty \equiv \langle a_\infty^\dagger a_\infty\rangle = \int_0^\infty \mathrm{d}k\,\rho(k)[|\alpha(k)|^2 n(k) + |\beta(-k)|^2(n(k)+1)]. \tag{4.9}$$

Restricting ourselves to a Markovian limit $(\gamma\tau\to 0)$, we obtain from (4.4), (2.10), (2.11) and (2.13)

$$|\alpha(k)|^2 = \left(\frac{\omega+\Omega}{2\Omega}\right)^2[(k-\Omega)^2+\gamma^2]^{-1} + \left(\frac{\omega-\Omega}{2\Omega}\right)^2[(k+\Omega)^2+\gamma^2]^{-1}$$
$$-\frac{|D|^2}{2\Omega^2}[(k^2-\Omega^2)+\gamma^2][(k+\Omega)^2+\gamma^2]^{-1}[(k-\Omega)^2+\gamma^2]^{-1} \tag{4.10}$$

$$|\beta(-k)|^2 = \frac{|D|^2}{4\Omega^2}[(k-\Omega)^2+\gamma^2]^{-1} + \frac{|D|^2}{4\Omega^2}[(k+\Omega)^2+\gamma^2]^{-1}$$
$$-\frac{|D|^2}{2\Omega^2}[k^2-\Omega^2+\gamma^2][(k+\Omega)^2+\gamma^2]^{-1}[(k-\Omega)^2+\gamma^2]^{-1}. \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9) and passing to the additional limit $\gamma\to 0$, we obtain

$$N_\infty = \frac{\omega+\Omega}{2\Omega}\,n(\Omega) + \frac{\omega-\Omega}{2\Omega}\,(n(\Omega)+1). \tag{4.12}$$

In deriving (4.12) we made use of the identity

$$\left.\frac{\varepsilon}{x^2+\varepsilon^2}\right|_{\varepsilon\to 0} = \pi\delta(x). \tag{4.13}$$

Notice that at the Markovian and weak dissipation limit, the system number density is independent of the undetermined function $\sigma_0(x)$. The equilibrium number density (4.12) corresponds to the Bose distribution for the Bogoliubov transformed operator $A(t)$.

For the Hermitian noise the KMS condition can be similarly proved. We just quote the result:

$$G^<(t) = \int_{-\infty}^\infty \mathrm{d}k\,\rho(|k|)[\theta(k)n(k) + \theta(-k)(n(|k|+1)]|\mu(k)+\nu(k)|^2\,\mathrm{e}^{-\mathrm{i}kt} \tag{4.14}$$

$$G^>(t) = \int_{-\infty}^\infty \mathrm{d}k\,\rho(|k|)[\theta(k)(n(k)+1) + \theta(-k)n(|k|)]|\mu(k)+\nu(k)|^2\,\mathrm{e}^{-\mathrm{i}kt}. \tag{4.15}$$

This again leads to the KMS condition. The equilibrium system number density is

$$N_\infty = \int_{-\infty}^\infty \mathrm{d}k\,\rho(|k|)[\theta(k)n(k) + \theta(-k)(n(|k|)+1)]|\mu(k)+\nu(k)|^2. \tag{4.16}$$

The Markovian $(\gamma\tau\to 0)$ and small dissipation limit again leads to the result (4.12).

5. Conclusion

In this paper we described the approach to a thermal equilibrium of a general quadratic system coupled to a heat bath, characterised either by a non-Hermitian or a Hermitian noise operator. This dissipation kernel is taken to be memory dependent. A set of dispersion relations for the bath distribution function is derived. These dispersion relations may be viewed as fluctuation dissipation theorems relating the dissipation kernel $\gamma(t)$ to the frequency mode distribution $\rho(k)$ characterising the noise operator. Close to the Markovian limit the distribution function is solved explicitly in a perturbative way. For a non-Hermitian noise operator our result for the bath frequency mode distribution function $\rho(k)$, when restricted to the case $D \sim 0$ and $\gamma\tau \sim 0$, agrees with that of Streater (1982). In the case of a Hermitian noise operator, however, $\rho(k) \sim k$. This agrees with the previous results (Hasegawa *et al* 1985, de Smedt *et al* 1988), which are special cases of the present example.

The KMS condition on the two-point functions is obtained in the limit when T is much greater than a typical timescale of the dissipation kernel. This describes the system achieving a thermal equilibrium with the heat bath at a temperature β^{-1}, and may be viewed as a check on the validity of the framework of the quantum Langevin equation. The equilibrium number density is also obtained.

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